Game & Strategies in Type Theory

Peio Borthelle, USMB, Chambéry, France
w/ Tom Hirschowitz & Guilhem Jaber & Yannick Zakowski

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Motivations

Goal

Study higher-order programming languages with effects such as non-termination starting from their operational semantics.

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Contextual Equivalence

"a and b are observationally indistinguishable"

$$a \simeq_{\mathrm{ctx}} b \quad := \quad \forall E,\, E[a] \simeq_{\mathrm{op}} E[b]$$

Program Equivalence

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Contextual Equivalence

"a and b are observationally indistinguishable"

$$a \simeq_{\mathrm{ctx}} b \quad := \quad \forall E,\, E[a] \simeq_{\mathrm{op}} E[b]$$

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We want an easier-to-check $\simeq_M$ such that

$$a \simeq_M b \quad \Longrightarrow \quad a \simeq_{\mathrm{ctx}} b$$

Operational Game Semantics

Trace Semantics

- Sequences of observations of an execution.
- "perform an effect", "call a free variable", "return a value"

An example: OGS

- In the spirit of process calculi: keep the labels first-order.
- Computations[1] are hidden as fresh variables.
- We don't observe full values but only patterns.

[1]functions, thunks … CBPV negative types

Our Contribution

- A generic account of Operational Game Semantics.
- Implemented and proved correct in Coq.

Formalization

Given an evaluator "$\mathrm{term} \to \mathcal{D}(\mathrm{nf})$":

- Construct the OGS LTS ;
- Show it correct for contextual equivalence.

Crux of the proof

$$[\![E[a]]\!] \approx_M [\![E]\!] \parallel [\![a]\!]$$

assuming the evaluator verifies:

$$\begin{tikzcd}
E[a] \arrow[r, mapsto, "*"] \arrow[d, mapsto, "*", "\text{“active”}"'] & v \\
E[w] \arrow[r, mapsto, "\text{“sync”}"'] & b \arrow[u, mapsto, "*"', "\text{“passive”}"]
\end{tikzcd}$$

Technical Choices

Intensional representations

- LTS more intensional than prefix-closed set of traces
- coalgebraic LTS compute more than relational LTS

$\Rightarrow$ guarded coinduction in Type using negative records

$\Rightarrow$ coinduction-up-to in Prop using Coq-Coinduction[2]

Rigid structures

- dependent variant of interaction trees[3]
- well-typed & well-scoped variables

$\Rightarrow$ dependent programming in Coq using Coq-Equations[4]

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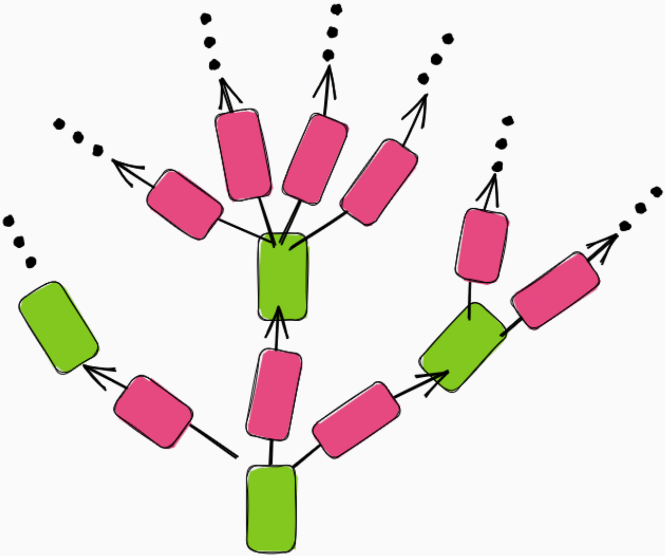
[2] by Damien Pous

[3] original by Li-Yao Xia *et al.*

[4] by Matthieu Sozeau

What are game rules?

Two Sets



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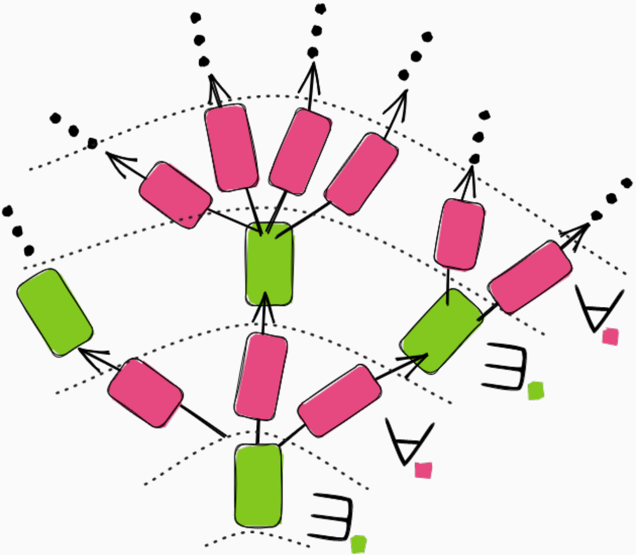

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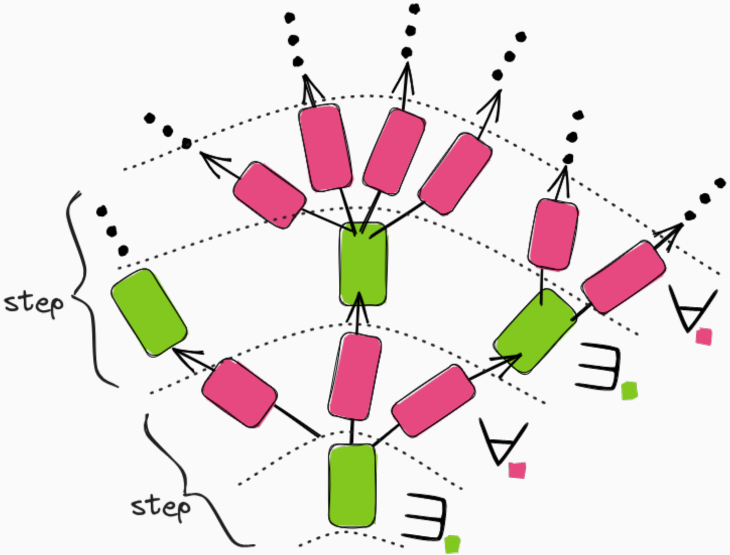

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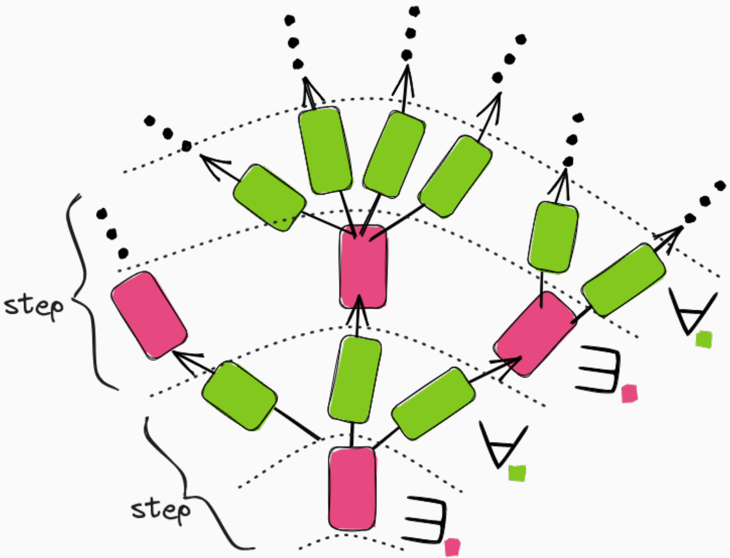

Simple Strategies, Formally

$$Step_P, Step_O : \mathbf{SET} \to \mathbf{SET}$$

$$Step_P\ X \coloneqq M_P \times (M_O \to X) \qquad \text{player step}$$

$$Step_O\ X \coloneqq M_O \times (M_P \to X) \qquad \text{opponent step}$$

Building Blocks

$$Act_M, Pas_M : \mathbf{SET} \to \mathbf{SET}$$

$$Act_M\ X \coloneqq M \times X \qquad \text{active half-step}$$

$$Pas_M\ X \coloneqq M \to X \qquad \text{passive half-step}$$

$$sync_M : Act_M\ X \times Pas_M\ Y \to X \times Y \qquad \text{interaction law}$$

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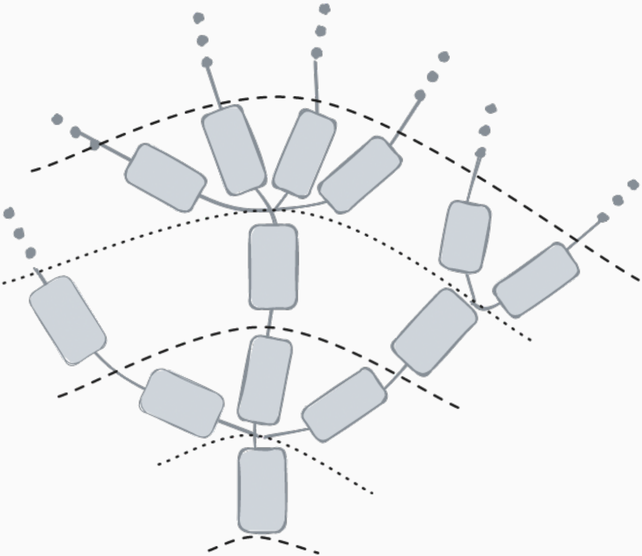
Pretty nice, but not very expressive.

More *Precise* Strategies



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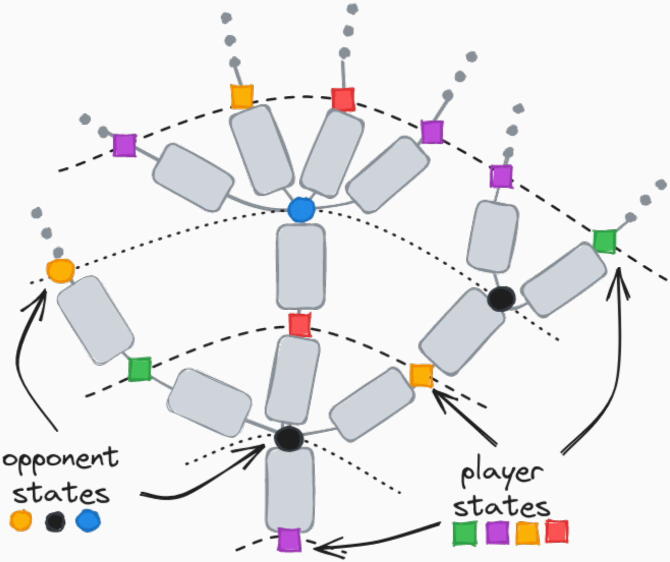
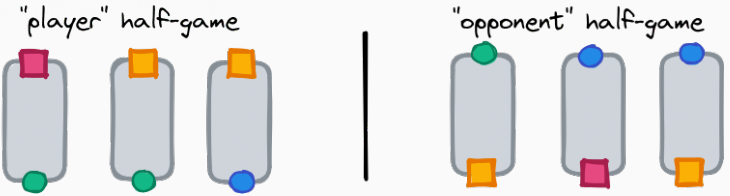


Two *Bi-Indexed Sets*[5]



$$HalfGame\ (I\ J : \mathbf{SET}) : \mathbf{SET} := \begin{cases} M : I \to \mathbf{SET} \\ f_i : S\ i \to J \end{cases}$$

$$Act, Pas : (J \to \mathbf{SET}) \to (I \to \mathbf{SET})$$

$$Act\ X\ i := (s : M\ i) \times X\ (f_i\ s) \qquad \text{active half-step}$$

$$Pas\ X\ i := (s : M\ i) \to X\ (f_i\ s) \qquad \text{passive half-step}$$

$$sync : \Sigma_i (Act\ X\ i \times Pas\ Y\ i) \to \Sigma_j (X\ j \times Y\ j) \qquad \text{interaction law}$$

[5] See also Paul Levy & Sam Staton: Transition systems over games

From pairs of "half-games" to indexed containers

$$Game\,(I\,J : \mathbf{SET}) : \mathbf{SET} := \begin{cases} P : HalfGame\ I\ J \\ O : HalfGame\ J\ I \end{cases}$$

- $Act_P \circ Pas_O : PolyEndo(I \to \mathbf{SET})$, player point of view
- $Act_O \circ Pas_P : PolyEndo(J \to \mathbf{SET})$, opponent point of view

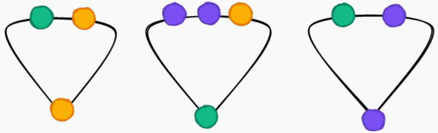
Moral: containers loose information

- In containers, implicitely $J = (i : I) \times M_P$.
- Let's cut containers in half!

Tree constructions

Classical container fixpoints

Given the “tiles”, how are we allowed to combine them?



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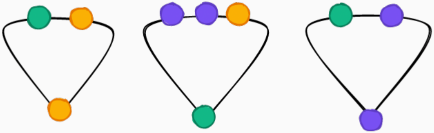


μ: inductive trees

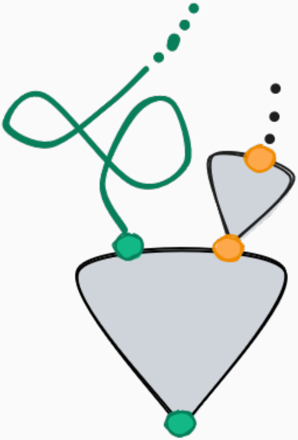
- “any combination of finite depth”
- “any algebra of the step-functor”

ν: coinductive trees

- “any combination”
- “any coalgebra of the step-functor”

Fixpoints, continued

But we also need these!



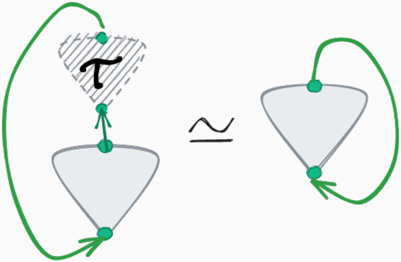
Interaction Trees! (no fancy greek letter for this fixpoint yet)

- “any combination with arbitrary loops”
- “any iterative algebra of the step-functor”
- “any coalgebra of ‘Step + Id’ modulo weak bisimilarity”

[6]See extensive work by Stefan Milius

Indexed Interaction Trees

$$ITree\,(X : I \to SET) := \nu A.(i \mapsto \underbrace{X\,i}_{\text{return}} + \overbrace{A\,i}^{\text{silent step}} + \underbrace{[\![G]\!]\;A\,i}_{\text{play a move}})$$



Weak bisimilarity skips over any finite number of τ nodes

$\Rightarrow$ Mixed inductive-coinductive,

but not a problem for Coq-Coinduction!

Conclusion

Contributions

- Formalized generic soundness theorem for operational game semantics.
- New datastructure: indexed interaction trees.

Ongoing & future work

- Clarify the categorical structure of "games" and "half-games".
- Fully formalized game semantics.

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Thanks for listening!

Composition of dual strategies